Fisher Transfer Entropy: Quantifying the gain in transient sensitivity

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We introduce a novel measure, Fisher Transfer Entropy, which quantifies a gain in sensitivity to a control parameter of a state transition, in the context of another observable source. The new measure captures both transient and contextual qualities of transfer entropy and the sensitivity characteristics of Fisher information. Fisher Transfer Entropy is exemplified for a ferromagnetic 2d lattice Ising model with Glauber dynamics, and is shown to diverge at the critical point.

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I. INTRODUCTION

Interactions within a complex dynamical system often induce intricate statistical regularities and rich information flows. Formalizing these regularities and flows in a dynamic distributed setting is a subject of Information Thermodynamics — an emerging field combining approaches from Information Theory, Statistical Estimation Theory, Complex Dynamical Systems, and Statistical Mechanics in an attempt to systematically and information-theoretically quantify spatiotemporal patterns on both a global and a local scale. This in turn enables a comprehensive comparative analysis of system dynamics across diverse physical, computational, biological and technological domains. Furthermore, discovering patterns of information thermodynamics within the system is crucial in identifying critical regimes and phase transitions, and providing efficient means for an accurate forecasting and precise control of system behavior.

One of the key challenges of Information Thermodynamics [1] is a lack of rigorous characterization of a dynamic balance between various information flows in the vicinity of phase transitions. An adequate information-theoretic framework for critical, edge-of-chaos, phenomena is yet to be developed. On the one hand, it is conjectured that at the edge of chaos the distributed computation, intrinsic to complex dynamics, maintains a balance between high information storage, information transfer, and synergistic information (or novelty generation). For example, transfer entropy [2], characterizing the communication aspect of computation, is known to peak near critical regimes [3, 4], while Fisher information [5] is known to peak at phase transitions [6]. On the other hand, it remains unclear how such dynamic balance is related to physical fluxes which are observed and studied during phase transitions. This work is motivated by the need to develop a new measure which shares properties of both transfer entropy and Fisher information, and apply it to a well-understood physical model.

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Computation-theoretically, transfer entropy was shown to capture one of the three elements of distributed computation: communication from system Y to system X [7, 8]. Transfer entropy was observed to be locally maximized in coherent propagating spatiotemporal structures within cellular automata (i.e., gliders) [7], and self-organizing swarms (cascading waves of motions) [9]. In another context, transfer entropy was found to be high while a system of coupled oscillators was beginning to synchronize, followed by a decline from the global maximum as the system was moving towards a synchronized state [10].

Thermodynamically, transfer entropy was found to be proportional to the external entropy production by the system X in the context of Y, due to irreversibility (e.g., heat flux) [11, 12]. In addition, maxima of transfer entropy were observed to be related to critical behaviour, e.g., average transfer entropy was observed to maximize on the chaotic side of the critical regime within random Boolean networks [3]. Furthermore, in a ferromagnetic 2d lattice Ising model with Glauber dynamics, (collective) transfer entropy was analytically shown to peak on the disordered side of the phase transition [4].

Elements of the Fisher information matrix were explicitly related to gradients of the corresponding order parameters [6], providing another important connection between information-theoretic and thermodynamic interpretations of critical behaviour. It is obvious, however, that transfer entropy and Fisher information reflect on quite different aspects of the dynamics. Information-theoretically, transfer entropy is centred on information dynamics during state *transitions* in the *context* of another source, while Fisher information quantifies the amount of information in an observable variable about a parameter, and thus estimating *sensitivity* to changes in the parameter. Thermodynamically, transfer entropy is proportional to the external entropy produced by a system during a transition, while Fisher information is proportional to the gradient of an order parameter, diverging when the system approaches a critical point. Under certain conditions these two measures can be explicitly related [13], i.e., in isothermal systems near thermodynamic equilibrium, the gradient of the average transfer entropy is shown to be dynamically related to Fisher information and the curvature of system's entropy. In other words, "predictability" of computation (transfer entropy) is explicitly connected to its "sensitivity" (Fisher information) and "uncertainty" (thermodynamic entropy).

There are other results relating Fisher information with various entropy measures (for example, [14–20]), as well as showing that Fisher Information provides a variational principle using which it is possible to derive, under suitable constraints, several fundamental physical laws in equilibrium and non-equilibrium thermodynamics, e.g. [16, 21–24], and also relating Fisher Information to synaptic plasticity and complexity of neural networks [25].

However, the main motivation of this study is a desire to meaningfully *combine* Fisher information and transfer entropy, in order to capture both *transient* and *contextual* qualities of transfer entropy and *sensitivity* characteristics of Fisher information. This may ultimately reveal fundamental connections between information dynamics at critical points.

We introduce here a novel measure, Fisher Transfer Entropy, which aims to quantify a gain in sensitivity to a control parameter, obtained during a (state) transition of an observable random variable, in the context of another observable random variable. The approach is then applied to a kinetic Ising model where we initially derive Fisher Information, showing analytically its divergence at the critical point, followed by a derivation and analysis of Fisher Transfer Entropy.

II. TECHNICAL PRELIMINARIES

Transfer entropy is a Shannon information-theoretic quantity [2] which measures a directed relationship between two, possibly coupled, time-series processes Y and X, by detecting asymmetry in their interactions. Specifically, the transfer entropy $T_{Y\to X}$ measures the average amount of information that states $\mathbf{y_n}$ at time n of the *source* time-series process Y provide about the next values $\mathbf{x_{n+1}}$ of the *destination* time-series process X, in the context of the previous state $\mathbf{x_n}$ of the destination process:

$$T_{Y\to X} = \left\langle \log_2 \frac{p(\mathbf{x_{n+1}} \mid \mathbf{x_n}, \mathbf{y_n})}{p(\mathbf{x_{n+1}} \mid \mathbf{x_n})} \right\rangle. \quad (1)$$

To be clear, *state* here refers to the underlying dynamical state of a process. For a time-series process X this is generally represented by Takens' *embedding vectors* [26] $\mathbf{x}_n^{(k,\tau)} = \{x_{n-(k-1)\tau}, \dots, x_{n-\tau}, x_n\}$, with *embedding dimension* k and embedding delay τ. In a thermodynamic setting, a set of thermodynamic state variables fulfils the same role.

The transfer entropy $T_{Y\to X}$, defined by (1), is a conditional mutual information [27] between $\mathbf{Y_n}$ and $\mathbf{X_{n+1}}$ given $\mathbf{X_n}$. Informally, it helps to answer the question *"if I know the state of the source, how much does that help to predict the state transition of the destination?"*.

Fisher information and the Fisher information matrix are well known in statistical estimation theory. Fisher information [5] is a measure for the amount of information that an observable random variable $\mathscr{X}$ provides about an unknown parameter θ, upon which the likelihood function of θ depends. Let $p(x|\theta)$ be the likelihood function of θ given the observations x. Then, Fisher information can be written as:

$$F(\theta) = \mathrm{E}\left[\left(\frac{\partial}{\partial \theta} \ln p(x|\theta) \right)^2 \middle| \theta \right] \quad (2)$$

$$= \int_x \left(\frac{\partial \ln(p(x|\theta))}{\partial \theta} \right)^2 p(x|\theta) dx \quad (3)$$

$$= \int_x \left(\frac{\partial p(x|\theta)}{\partial \theta} \right)^2 \frac{1}{p(x|\theta)} dx, \quad (4)$$

where $E[\dots|\theta]$ denotes the conditional expectation over values for $x \in \mathscr{X}$ with respect to the probability function $p(x|\theta)$ given θ. Thus, Fisher information is not a function of a particular observation, since the random variable $\mathscr{X}$ has been averaged out.

The discrete form of Fisher information is:

$$F(\theta) = \sum_x p(x|\theta) \left(\frac{\partial \ln p(x|\theta)}{\partial \theta} \right)^2, \quad (5)$$

$$= \sum_x \frac{1}{p(x|\theta)} \left(\frac{\partial p(x|\theta)}{\partial \theta} \right)^2. \quad (6)$$

In this case, $p(x)$ is a discrete probability distribution function, such that $x \in \{x_1, \dots, x_D\}$, where D is the alphabet size or number of potential values for the variable $\mathscr{X}$.

Furthermore, the $n \times n$ Fisher information matrix is defined for several parameters $\theta = [\theta_1, \theta_2, \dots, \theta_n]^{\mathrm{T}}$, as follows

$$F_{ij}(\theta) = \mathrm{E}\left[\left(\frac{\partial}{\partial \theta_i} \ln p(x|\theta) \right) \left(\frac{\partial}{\partial \theta_j} \ln p(x|\theta) \right) \middle| \theta \right]. \quad (7)$$

Statistical-mechanics models typically deal with Gibbs measures, defined for a physical system in an equilibrium with a large thermal reservoir, as follows:

$$p(x|\theta) = \frac{1}{Z(\theta)} e^{-\sum_i \theta^i X_i(x)} = \frac{1}{Z(\theta)} e^{-\beta H(x,\theta)}, \quad (8)$$

where: the configuration variable x varies over the configuration space; the set $\{\theta^i\}$ includes time-dependent thermodynamic variables (e.g., inverse temperature, pressure, magnetic field, chemical potential, etc.); and the time-independent functions $X_i(x)$ are collective variables which determine the form of action. The system's Hamiltonian captures the total energy at x: $\beta H(x,\theta) = \sum_i \theta^i X_i(x)$, with $\beta = 1/k_B T$ being the inverse temperature (T) of the environment in natural units, and k_B denoting the Boltzmann constant, and $Z(\theta) = \sum_x e^{-\beta H(x,\theta)}$ is the partition function [28, 29].

Several previous studies [28–31] reported that Fisher information matrix provides a Riemannian metric (more precisely, the Fisher–Rao metric) for the manifold of thermodynamic states. For instance, it was suggested that the scalar curvature $\mathfrak{R}$ of the thermodynamic metric tensor $g_{ij}(\theta) = F_{ij}(\theta)$ measures the complexity of the system [30].

Fisher information is also explicitly related to the gradient of the corresponding order parameter(s) [6]:

$$F_{ij}(\theta) = \beta \frac{\partial \phi^i}{\partial \theta^j}, \quad (9)$$

where the order parameter ϕ^i is a negative derivative of thermodynamic potential $G = -k_B T \ln Z$ over some thermodynamic variable θ^i, i.e. $\phi^i = -\frac{\partial G}{\partial \theta^i}$. The order parameter is known to be related to the mean value of the corresponding collective variable X_i [6, 32]:

$$\phi^i = -k_B T \langle X_i \rangle . \tag{10}$$

During a second-order phase transition the order parameter changes continuously when an independent variable is varied, going to zero at the critical point, while Fisher information exhibits divergence. In finite-size computational studies the divergence can be approximated by maximization of Fisher information [33, 34]. As pointed out by [6], not only does this avoid the issue of identifying order parameters, but also provides a natural interpretation of localizing the critical point where the observed variable is most sensitive to the control parameter(s) / thermodynamic variable(s) (an interpretation applicable in both infinite and finite systems).

The following general relationship [6, 35]

$$F_{ij}(\theta) = \langle (X_i(x) - \langle X_i \rangle)\,(X_j(x) - \langle X_j \rangle) \rangle \tag{11}$$

gives the covariance matrix between the collective variables X_i and X_j. Thus, the Fisher information F_{ij} can be seen to measure "the size of fluctuations about equilibrium" of the collective variables X_i and X_j [35].

Using this general expression, one may consider a generic case of a d-dimensional Ising-type magnetic model with a probability density expressible in the form of equation (8) [28]. For this model, Brody and Rivier [28] have shown that critical behaviour of thermodynamic quantities can be analyzed in terms of the reduced temperature $t = T/T_c - 1$, leading to the general expression [28]:

$$F_{ij}(\theta) \sim \begin{pmatrix} |t|^{-\alpha} & |t|^{b-1} \\ |t|^{b-1} & |t|^{-\gamma} \end{pmatrix} . \tag{12}$$

One may demonstrate divergence of certain elements of the Fisher information matrix at the critical point (where $T \to T_c$ and $t \to 0$) for specific cases of d and the corresponding values of critical exponents (e.g., for the 3-dimensional Ising model all matrix elements diverge).

III. FISHER INFORMATION IN A KINETIC ISING MODEL

We consider an isotropic ferromagnetic 2d lattice Ising model of size $N = L \times L$, with no external field. If the system is in state $\mathbf{s} = s_1, \ldots, s_N, s_i \in \{+1, -1\}$, then the Hamiltonian is given by

$$H(\mathbf{s}) = -\sum_{\langle i,j \rangle} s_i s_j , \tag{13}$$

where $\langle i, j \rangle$ denotes a sum over the $2N$ unique pairs of lattice neighbours, bold type $\mathbf{s}$ denotes a state vector of spins and normal/lower case Greek type s_i denotes individual ± 1 spin of site i. In the following, we also use capitals $\mathbf{S}, S_i$ to denote random variables, and σ to denote a specific spin value ± 1.

For the kinetic model we consider discrete-time Glauber spin-flip dynamics [36]: at each time step a site i is selected uniformly at random and its spin flipped with probability

$$P_i(\mathbf{s}) = \left[1 + e^{\beta \Delta H_i(\mathbf{s})}\right]^{-1} , \tag{14}$$

where $\Delta H_i(\mathbf{s}) = H(s^i) - H(\mathbf{s})$ is the energy difference between the spin-flipped and original state, and units are considered normalized so that Boltzmann constant k_B is 1. Here a superscript i denotes flipping the i-th spin.

The following analytical expression, obtained by Onsager, is well-known for the magnetization as a function of temperature [37]:

$$M = \pm \left(1 - \left[\sinh\left(\log(1+\sqrt{2})\frac{T_c}{T}\right)\right]^{-4}\right)^{\frac{1}{8}} \tag{15}$$

$$= \pm \left[1 - \sinh^{-4}(2\beta)\right]^{1/8} \tag{16}$$

for $T < T_c$, where $T_c = \frac{2}{\log(1+\sqrt{2})}$; and $M = 0$ for $T \geq T_c$.

Fisher information of the spin at a specific site i can then be analytically derived as a function of inverse temperature β (see Appendix VII for details):

$$F_i(\beta) = \frac{\coth^2(2\beta)\sinh^{-8}(2\beta)}{\left(1 - \sinh^{-4}(2\beta)\right)^{7/4}\left(1 - \left(1 - \sinh^{-4}(2\beta)\right)^{1/4}\right)} \tag{17}$$

As shown in Appendix VII, Fisher information of each spin diverges at critical temperature, as expected. In other words, as $T \to T_c$ from below, i.e., $\beta \to \beta_c$ from above, where $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$, Fisher information $F_i(\beta) \to \infty$.

For the disordered case $T \geq T_c$, the magnetization $M = 0$ and the probabilities $p(S_i = \sigma) = \frac{1}{2}$ are constant in the thermodynamic limit according to (30), specified in Appendix VII, resulting in a trivial result: $F_i(\beta) = 0$.

IV. FISHER TRANSFER ENTROPY

We introduce here a novel measure, Fisher Transfer Entropy, which quantifies a sensitivity gain from a (state) transition of an observable random variable X, in the context of another observable random variable Y, to an unknown parameter θ:

$$F_{Y \to X}(\theta) = F_{X_{n+1}|X_n,Y_n}(\theta) - F_{X_{n+1}|X_n}(\theta) , \tag{18}$$

where

$$F_{X_{n+1}|X_n}(\theta) = \int_{x_n}\int_{x_{n+1}} \left(\frac{\partial \ln(p(x_{n+1}|x_n,\theta))}{\partial\theta}\right)^2 p(x_{n+1}|x_n,\theta)\; dx_n\; dx_{n+1} \tag{19}$$

$$= \int_{x_n}\int_{x_{n+1}} \left(\frac{\partial p(x_{n+1}|x_n,\theta)}{\partial\theta}\right)^2 \frac{1}{p(x_{n+1}|x_n,\theta)}\; dx_n\; dx_{n+1}\,, \tag{20}$$

and

$$F_{X_{n+1}|X_n,Y_n}(\theta) = \int_{x_n}\int_{y_n}\int_{x_{n+1}} \left(\frac{\partial \ln(p(x_{n+1}|x_n,y_n,\theta))}{\partial\theta}\right)^2 p(x_{n+1}|x_n,y_n,\theta)\; dx_n\; dy_n\; dx_{n+1} \tag{21}$$

$$= \int_{x_n}\int_{y_n}\int_{x_{n+1}} \left(\frac{\partial p(x_{n+1}|x_n,y_n,\theta)}{\partial\theta}\right)^2 \frac{1}{p(x_{n+1}|x_n,y_n,\theta)}\; dx_n\; dy_n\; dx_{n+1}\,. \tag{22}$$

The term $F_{X_{n+1}|X_n}(\theta)$, specified in equations (19)–(20), quantifies the transient (or dynamic) sensitivity of a state transition from X_n to X_{n+1} of the observable random variable X, to parameter θ. The term $F_{X_{n+1}|X_n,Y_n}(\theta)$, given by equations (21)–(22), accounts for the transient sensitivity to θ of a state transition from X_n to X_{n+1} given Y_n: that is, the transition of the variable X in the context of variable Y. Hence, the resulting difference between the terms, $F_{Y\to X}(\theta)$, captured by (18), measures the gain in transient sensitivity when variable Y is accounted for in the transition from X_n to X_{n+1}. That is, if variables X and Y are independent, and $F_{X_{n+1}|X_n}(\theta) = F_{X_{n+1}|X_n,Y_n}(\theta)$, there is no transient sensitivity gain: $F_{Y\to X}(\theta) = 0$. Unlike Fisher information, the introduced Fisher Transfer Entropy and its terms measure (a gain in) transient, or dynamic, sensitivity for a state transition, rather than (a gain in) the amount of information contained in different variables.

The terms $F_{X_{n+1}|X_n}(\theta)$ and $F_{X_{n+1}|X_n,Y_n}(\theta)$ can be represented using the Chain Rule for Fisher Information [38]:

$$F_{A,B}(\theta) = F_A(\theta) + F_{B|A}(\theta)\,, \tag{23}$$

and so, in general, Fisher Transfer Entropy can be decomposed as follows:

$$F_{Y\to X}(\theta) = F_{X_{n+1},X_n,Y_n}(\theta) - F_{X_n,Y_n}(\theta) - \left[F_{X_{n+1},X_n}(\theta) - F_{X_n}(\theta)\right]\,. \tag{24}$$

To obtain a discrete form for FTE, one simply applies the discrete form for Fisher Information to each term in equation (24) above (see specific examples in Appendix VIII).

V. FISHER TRANSFER ENTROPY IN A KINETIC ISING MODEL

In this section we derive Fisher Transfer Entropy (FTE) for a kinetic Ising model, focusing only on the disordered phase, i.e. the simpler case of $T \geq T_c$, as $\beta \to \beta_c$ *from below*. In this phase Fisher information is trivially zero: $F_i(\beta) = 0$, as established in the previous section. Henceforth, X is the random variable associated with a given lattice site, and Y is the random variable representing one of its lattice neighbours.

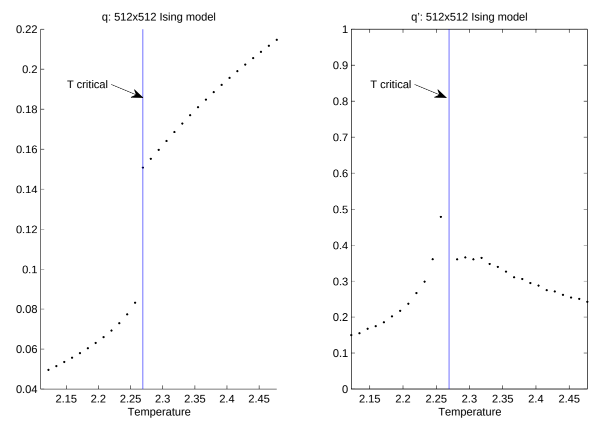

FIG. 1. Left: q as a function of T. Right: $\frac{\partial q}{\partial T}$ as a function of T. Results obtained after 10,000 runs of Ising model of size $N = 512$.

The first term in the FTE is given as follows (see Appendix VIII):

$$F_{X_{n+1}|X_n}(\beta) = \frac{4}{q(N-2q)}\left(\frac{\partial q}{\partial\beta}\right)^2, \tag{25}$$

where, in the thermodynamic limit,

$$q = \frac{1}{2}\langle P_i(\mathbf{S})\rangle \tag{26}$$

with i being the site index, $\mathbf{S}$ the stochastic variable representing the system (lattice) vector, and a state vector $\mathbf{s}$ representing an instance of $\mathbf{S}$. $P_i(\mathbf{S})$ is the flipping probability of spin S_i in a given spin configuration $\mathbf{S}$ (n.b. equation (14)). In general, q is a function of β or T. Results of a numerical estimation of q are shown in Fig. 1 (left).

A numerical estimation of $\frac{\partial q}{\partial T}$ shows, cf. Fig. 1 (right), that it is positive at critical temperature, and the estimates increase as the differences ΔT decrease, so that $\frac{\partial q}{\partial T} \to \infty$. Since $\frac{\partial}{\partial\beta} = -T^2\frac{\partial}{\partial T}$, we can conclude that $\frac{\partial q}{\partial\beta} \to -\infty$ at critical point. As $1 > q > 0$, the total transient sensitivity of a state transition from X_n to X_{n+1} diverges at β_c: $NF_{X_{n+1}|X_n}(\beta) \to \infty$.

The second term in the FTE is given by following expression (obtained in Appendix VIII):

$$F_{X_{n+1}|X_n,Y_n}(\beta) = 4\left(\frac{\left(\frac{\partial q}{\partial \beta}U_- - q\frac{\partial U_-}{\partial \beta}\right)^2}{qU_-^2\left(NU_- - 2q\right)} + \frac{\left(\frac{\partial q}{\partial \beta}U_+ - q\frac{\partial U_+}{\partial \beta}\right)^2}{qU_+^2\left(NU_+ - 2q\right)}\right), \quad (27)$$

where U is the internal energy (as a function of β, see equation (46)), and $U_- = 1 - \frac{1}{2}U$ and $U_+ = 1 + \frac{1}{2}U$. It has been established that $\frac{\partial U}{\partial \beta} \to -\infty$ logarithmically as $\beta \to \beta_c$ from below [39]. This, together with numerical observation $\frac{\partial q}{\partial \beta} \to -\infty$, yields divergence of the total transient sensitivity of a state transition from X_n to X_{n+1} in context of Y_n at critical point as well: $NF_{X_{n+1}|X_n,Y_n}(\beta) \to \infty$.

The difference between (27) and (25) yields the desired Fisher Transfer Entropy defined by (18), as the gain in transient sensitivity. In the thermodynamic limit, we obtain

$$F_{Y\to X}(\beta) \to \frac{A_{q,U}\left(\frac{\partial q}{\partial \beta}\right)^2 + B_U \frac{\partial q}{\partial \beta}\frac{\partial U}{\partial \beta} + C_{q,U}\left(\frac{\partial U}{\partial \beta}\right)^2}{N}, \quad (28)$$

where coefficients $A_{q,U}$, B_U and $C_{q,U}$ are positive at the critical point (see Appendix VIII).

Given logarithmic divergence $\frac{\partial U}{\partial \beta} \to -\infty$ and the results of numerical estimations suggesting that $\frac{\partial q}{\partial \beta} \to -\infty$, being negative in any case, and thus making the product $\frac{\partial q}{\partial \beta}\frac{\partial U}{\partial \beta}$ always positive, we obtain that the total Fisher Transfer Entropy diverges, $N\,F_{Y\to X}(\beta) \to \infty$ at the critical point, as $\beta \to \beta_c$ from below.

Since $\frac{\partial}{\partial \beta} = -T^2\frac{\partial}{\partial T}$, we also obtain

$$N\,F_{Y\to X}(T) \to T^4\left(A_{q,U}\left(\frac{\partial q}{\partial T}\right)^2 + B_U\frac{\partial q}{\partial T}\frac{\partial U}{\partial T} + C_{q,U}\left(\frac{\partial U}{\partial T}\right)^2\right), \quad (29)$$

and so, as $T \to T_c$ from above, the total Fisher Transfer Entropy diverges as well: $N\,F_{Y\to X}(T) \to \infty$.

In other words, at the critical regime, divergence of the transient sensitivity in context of the neighbors, i.e. $NF_{X_{n+1}|X_n,Y_n}(\beta)$, is faster than divergence of the transient sensitivity *per se*, $NF_{X_{n+1}|X_n}(\beta)$, and so the gain in transient sensitivity diverges overall. Interestingly, this can be contrasted with zero Fisher information: $F_i(\beta) = 0$, on the disordered side; highlighting that FTE reveals changes in dynamic sensitivity that Fisher information does not.

VI. DISCUSSION AND CONCLUSION

This study introduced *Fisher Transfer Entropy*, a measure which quantifies a gain in sensitivity to a control parameter. This gain is obtained during a state transition of an observable random variable X ("destination"), in the context of another observable random variable Y ("source"). The new measure combines several characteristics of two well-known measures: transfer entropy and Fisher information. It captures transient and contextual qualities of transfer entropy, as well as sensitivity of Fisher information. The "destination" variable *per se* may be insensitive to changes in some control parameter θ, resulting in zero Fisher information $F_X(\theta)$. Moreover, even a transition between the states of the "destination" variable may gain no sensitivity to the control parameter changes, with $F_{X_{n+1}|X_n}(\theta) = 0$. However, when such a transition occurs in context of some external influence, e.g., provided by "source" Y, the transient dynamics may become sensitive to changes in θ , with non-zero transient contextual sensitivity: $F_{X_{n+1}|X_n,Y_n}(\theta) \neq 0$. The gain in transient sensitivity is brought about by the source-destination interaction, which may be due to either direct influence or some indirect contextual contribution from the source.

It is well-known that non-zero transfer entropy does not necessarily mean that the source has a causal effect on the destination [40], and so the introduced Fisher Transfer Entropy is not intended to capture any sensitivity or gain in causal interactions between the variables. It does, nevertheless, capture the gain in transient sensitivity of the destination variable in presence of the source variable. Informally, Fisher Transfer Entropy refers to the amount of informational sensitivity that a source variable adds to the next state of a destination variable; i.e., addressing the question "if I know the state of the source, how much does that help in gaining sensitivity of the state transition of the destination, to changes in some control parameter?".

One may then pose the question, "In which situations would Fisher Transfer Entropy reveal interesting dynamics?" As pointed out in preceding paragraphs, the proposed measure is focussed on sensitivity of transient dynamics, in context of some external source *interacting* with the dynamics under the consideration. We can expect that such interactions exhibit some non-trivial dynamics in the vicinity of phase transitions, when variables are characterised by critical behaviour and so may be particularly sensitive to changes in the underlying control parameter. Moreover, the gain in the sensitivity within an interacting system may further characterise the strength and/or complexity of the interaction.

We applied the approach to a kinetic Ising model. Fisher Information was analytically shown to diverge at the critical point approaching from one side (cf. empirical results of another study confirming this derivation [41]), and staying zero on the other side. We followed with a detailed analysis of Fisher Transfer Entropy and demonstrated its divergence at the critical point, approaching from the same side where Fisher Information itself is actually zero. The reason for zero Fisher Information is that the opposite spin states are in balance on that side and remain insensitive to the temperature, whereas the interactions in the transient dynamics are sensitive to the temperature. Furthermore, the results show that sensitivity of transient dynamics diverges faster in presence of the interactions with the lattice neighbours of Ising model, yielding non-zero Fisher Transfer Entropy.

Collective transfer entropy was previously analytically shown to peak on the disordered side of the phase transition [4]. And so it remains an intriguing question whether collective Fisher Transfer Entropy, generalised to account for influences of all lattice neighbours, may also have a (post-critical) peak on the disordered side.

There are several other avenues for future research and applications. We believe that measuring Fisher Transfer Entropy may be particularly useful in systems with strong coupling and interacting components. For example, interactions within bipartite systems may undergo critical changes near or at phase transitions, and estimating the gain in transient sensitivity may reveal and/or characterise specific phase transitions driven by external influences. Similarly, many real-world complex networks are interdependent, and recent theoretical work on "networks formed from interdependent networks" suggests that when interdependencies are introduced, some well-known properties no longer hold (e.g., scale-free networks coupled with other networks lose their robustness to random failures [42]). Again, Fisher Transfer Entropy measured within such networks may identify salient pathways for critical information dynamics.

VII. APPENDIX A: FISHER INFORMATION

We follow [4] in specifying the distribution

$$p(S_i=\sigma) = \sum_s p(\mathbf{S}=s)p(S_i=\sigma|\mathbf{S}=s) \tag{30}$$

$$= \frac{1}{2}(1+\sigma\langle S_i\rangle) \;\to\; \frac{1}{2}(1+\sigma M) \tag{31}$$

as $N\to\infty$. Using thermodynamic limit for $T<T_c$ (16), we can rewrite the probability distribution as

$$p(S_i=\sigma) = \frac{1}{2}\left(1\pm\sigma\left[1-\sinh^{-4}(2\beta)\right]^{1/8}\right), \tag{32}$$

where the $\pm$ reflects the bifurcation in the system. This allows us to analytically derive Fisher information of a single site i as a function of inverse temperature β, by substituting $p(S_i=\sigma)$ into (6), and setting $\theta=\beta$, and $x=S_i$:

$$\begin{aligned}F_i(\beta) &= \sum_{S_i}\left(\frac{\partial p(S_i|\beta)}{\partial\beta}\right)^2\frac{1}{p(S_i|\beta)}\\ &= \frac{\coth^2(2\beta)\sinh^{-8}(2\beta)}{2\left(1-\sinh^{-4}(2\beta)\right)^{7/4}\left(1\pm\left(1-\sinh^{-4}(2\beta)\right)^{1/8}\right)}\\ &+ \frac{\coth^2(2\beta)\sinh^{-8}(2\beta)}{2\left(1-\sinh^{-4}(2\beta)\right)^{7/4}\left(1\mp\left(1-\sinh^{-4}(2\beta)\right)^{1/8}\right)}\\ &= \frac{\coth^2(2\beta)\sinh^{-8}(2\beta)}{\left(1-\sinh^{-4}(2\beta)\right)^{7/4}\left(1-\left(1-\sinh^{-4}(2\beta)\right)^{1/4}\right)}\end{aligned}$$

We can evaluate this expression as $T\to T_c$ from below, i.e., $\beta\to\beta_c$ from above, where $\beta_c=\frac{1}{2}\log(1+\sqrt{2})$. In doing so, we again follow [39], by specifically changing the sign of ε in the corresponding equations (15) and (18) [39]:

$$\sinh(2\beta) = 1+\sqrt{2}\varepsilon+\mathrm{O}(\varepsilon^2) \tag{33}$$

$$\coth(2\beta) = \sqrt{2}-\varepsilon+\mathrm{O}(\varepsilon^2) \tag{34}$$

It follows then that as $\beta\to\beta_c$ from above and $\varepsilon\to 0$, Fisher information $F_i(\beta)\to\infty$.

VIII. APPENDIX B: FISHER TRANSFER ENTROPY

The analysis presented here is limited to the simpler case of $T\geq T_c$, as $\beta\to\beta_c$ from below.

The relevant probability functions are given as follows [39]:

$$p(\sigma'|\sigma) = \begin{cases}1-\frac{1}{N}\frac{q}{p_\sigma} & \sigma'=\sigma\\ \frac{1}{N}\frac{q}{p_\sigma} & \sigma'=-\sigma\end{cases} \tag{35}$$

$$p(x_{n+1}|x_n) = \begin{cases}1-\frac{1}{N}\frac{q}{p(x_n)} & x_{n+1}=x_n\\ \frac{1}{N}\frac{q}{p(x_n)} & x_{n+1}=-x_n\end{cases} \tag{36}$$

where q is defined, in the thermodynamic limit, according to (26):

$$q = \frac{1}{2}\langle P_i(\mathbf{S})\rangle \tag{37}$$

with i being the site index, $\mathbf{S}$ the stochastic variable representing the system (lattice) vector, and a state vector $\mathbf{s}$ representing an instance of $\mathbf{S}$.

Next, we have [39]:

$$p(\sigma''|\sigma,\sigma') = \begin{cases}1-\frac{1}{N}\frac{q_{\sigma'}}{p_{\sigma\sigma'}} & \sigma''=\sigma\\ \frac{1}{N}\frac{q_{\sigma'}}{p_{\sigma\sigma'}} & \sigma''=-\sigma\end{cases} \tag{38}$$

where

$$q_{\sigma'} = \frac{1}{4}\left\langle P_i(\mathbf{S})+\sigma'\langle S_jP_i(\mathbf{S})\rangle\right\rangle \tag{39}$$

$$p(x_{n+1}|x_n,y_n) = \begin{cases}1-\frac{1}{N}\frac{q_y}{p(x_n,y_n)} & x_{n+1}=x_n\\ \frac{1}{N}\frac{q_y}{p(x_n,y_n)} & x_{n+1}=-x_n\end{cases} \tag{40}$$

where $q_y=q_{y_n}$ is defined according to (39).

Then, according to the discrete form of (20),

$$F_{X_{n+1}|X_n}(\beta) = \sum_{x_n,x_{n+1}}\left(\frac{\partial p(x_{n+1}|x_n,\beta)}{\partial\beta}\right)^2\frac{1}{p(x_{n+1}|x_n,\beta)}. \tag{41}$$

When symmetry is unbroken, in the thermodynamic limit the probability $p(x_n)=\frac{1}{2}$, and so (36) entails

$$\frac{\partial p(x_{n+1}|x_n,\beta)}{\partial\beta} = \begin{cases}-\frac{2}{N}\frac{\partial q}{\partial\beta} & x_{n+1}=x_n\\ \frac{2}{N}\frac{\partial q}{\partial\beta} & x_{n+1}=-x_n\end{cases} \tag{42}$$

Substituting (42) and (36) into (41) produces

$$F_{X_{n+1}|X_n}(\beta) = \frac{4}{q(N-2q)}\left(\frac{\partial q}{\partial \beta}\right)^2 . \quad (43)$$

Now we consider the discrete form of the more complicated term (22):

$$F_{X_{n+1}|X_n,Y_n}(\beta) = \sum_{x_n,y_n,x_{n+1}} \left(\frac{\partial p(x_{n+1}|x_n,y_n,\beta)}{\partial \beta}\right)^2 \frac{1}{p(x_{n+1}|x_n,y_n,\beta)} . \quad (44)$$

Reducing (39) for the case $T \geq T_c$, and noting that $\langle S_j P_i(\mathbf{S})\rangle = 0$ [4], results in $q_y = \frac{1}{2}q$. However, the conditional probability $p(x_{n+1}|x_n,y_n)$, specified by (40), does in general depend on β, creating several possibilities, dependent on spins of x_{n+1}, x_n, and y_n. According to [39], for infinite lattices,

$$p(S_i = \sigma, S_j = \sigma') \to \frac{1}{4}\left[1 + (\sigma+\sigma')M - \frac{1}{2}\sigma\sigma' U\right] , \quad (45)$$

where internal energy U is dependent on β, and is given by [4, 39]

$$U = -\coth 2\beta \left[1 + \frac{2}{\pi}(\kappa \sinh 2\beta - 1)K(\kappa)\right] \quad (46)$$

with

$$K(\kappa) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\kappa^2 \sin^2\phi}} . \quad (47)$$

Hence, the joint probability $p(x_n,y_n)$, which appears in denominators of (40), depends on β, as follows (we again use $M = 0$ for $T \geq T_c$):

$$p(x_n,y_n) = \frac{1}{4}\left[1 - \frac{1}{2}x_n y_n U\right] = \begin{cases} \frac{1}{4}\left[1-\frac{1}{2}U\right] & x_n = y_n \\ \frac{1}{4}\left[1+\frac{1}{2}U\right] & x_n = -y_n \end{cases} \quad (48)$$

We shall abbreviate $U_- = 1 - \frac{1}{2}U$ and $U_+ = 1 + \frac{1}{2}U$, so

$$p(x_n,y_n) = \begin{cases} \frac{1}{4}U_- & x_n = y_n \\ \frac{1}{4}U_+ & x_n = -y_n \end{cases} \quad (49)$$

and

$$\frac{\partial p(x_n,y_n)}{\partial \beta} = \begin{cases} \frac{1}{4}\frac{\partial U_-}{\partial \beta} & x_n = y_n \\ \frac{1}{4}\frac{\partial U_+}{\partial \beta} & x_n = -y_n \end{cases} \quad (50)$$

Taking derivative over β in (40), and using $q_y = \frac{1}{2}q$, produces

$$\frac{\partial p(x_{n+1}|x_n,y_n)}{\partial \beta} = \frac{\partial}{\partial \beta} \begin{cases} 1 - \frac{1}{2N}\frac{q}{p(x_n,y_n)} & x_{n+1} = x_n \\ \frac{1}{2N}\frac{q}{p(x_n,y_n)} & x_{n+1} = -x_n \end{cases} \quad (51)$$

$$= \frac{1}{2N} \begin{cases} -\left(\frac{\partial q}{\partial \beta}\frac{1}{p(x_n,y_n)} - \frac{\partial p(x_n,y_n)}{\partial \beta}\frac{q}{p^2(x_n,y_n)}\right) & x_{n+1} = x_n \\ \frac{\partial q}{\partial \beta}\frac{1}{p(x_n,y_n)} - \frac{\partial p(x_n,y_n)}{\partial \beta}\frac{q}{p^2(x_n,y_n)} & x_{n+1} = -x_n \end{cases} \quad (52)$$

So we have (for either case $x_{n+1} = \pm x_n$):

$$\left(\frac{\partial p(x_{n+1}|x_n,y_n)}{\partial \beta}\right)^2 = \frac{1}{4N^2}\left(\frac{\frac{\partial q}{\partial \beta}p(x_n,y_n) - q\frac{\partial p(x_n,y_n)}{\partial \beta}}{p^2(x_n,y_n)}\right)^2 \quad (53)$$

$$= \frac{4}{N^2} \begin{cases} \left(\frac{\frac{\partial q}{\partial \beta}U_- - q\frac{\partial U_-}{\partial \beta}}{U_-^2}\right)^2 & x_n = y_n \\ \left(\frac{\frac{\partial q}{\partial \beta}U_+ - q\frac{\partial U_+}{\partial \beta}}{U_+^2}\right)^2 & x_n = -y_n \end{cases} \quad (54)$$

where the last step used (49) and (50).

Substituting (54) and (40) into (44) and considering eight possible spin permutations of x_{n+1}, x_n, and y_n, yields

$$F_{X_{n+1}|X_n,Y_n}(\beta) = \frac{8}{N}\left(\frac{\left(\frac{\partial q}{\partial \beta}U_- - q\frac{\partial U_-}{\partial \beta}\right)^2}{U_-^3\,(NU_- - 2q)} + \frac{\left(\frac{\partial q}{\partial \beta}U_- - q\frac{\partial U_-}{\partial \beta}\right)^2}{2qU_-^3}\right) \quad (55)$$

$$+\frac{8}{N}\left(\frac{\left(\frac{\partial q}{\partial \beta}U_+ - q\frac{\partial U_+}{\partial \beta}\right)^2}{U_+^3\,(NU_+ - 2q)} + \frac{\left(\frac{\partial q}{\partial \beta}U_+ - q\frac{\partial U_+}{\partial \beta}\right)^2}{2qU_+^3}\right) \quad (56)$$

$$= 4\left(\frac{\left(\frac{\partial q}{\partial \beta}U_- - q\frac{\partial U_-}{\partial \beta}\right)^2}{qU_-^2\,(NU_- - 2q)} + \frac{\left(\frac{\partial q}{\partial \beta}U_+ - q\frac{\partial U_+}{\partial \beta}\right)^2}{qU_+^2\,(NU_+ - 2q)}\right) . \quad (57)$$

The difference between (57) and (43) yields the desired Fisher Transfer Entropy defined by (18). In the thermodynamic limit, we obtain

$$F_{Y\to X}(\beta) \to \frac{A_{q,U}\left(\frac{\partial q}{\partial\beta}\right)^2 + B_U \frac{\partial q}{\partial\beta}\frac{\partial U}{\partial\beta} + C_{q,U}\left(\frac{\partial U}{\partial\beta}\right)^2}{N}, \quad (58)$$

with coefficients $A_{q,U}$, B_U and $C_{q,U}$ given as follows:

$$A_{q,U} = \frac{4}{q}\left(\frac{1}{U_-} + \frac{1}{U_+} - 1\right) = \frac{4}{q\left(1 - \frac{U^2}{4}\right)} \quad (59)$$

$$B_U = 4\left(\frac{1}{U_-^2} - \frac{1}{U_+^2}\right) = \frac{8U}{\left(1 - \frac{U^2}{4}\right)^2} \quad (60)$$

$$C_{q,U} = q\left(\frac{1}{U_-^3} + \frac{1}{U_+^3}\right) = q\frac{2 + \frac{3U^2}{2}}{\left(1 - \frac{U^2}{4}\right)^3}. \quad (61)$$

The total Fisher Transfer Entropy within the system is then given by:

$$N\, F_{Y\to X}(\beta) \to A_{q,U}\left(\frac{\partial q}{\partial\beta}\right)^2 + B_U \frac{\partial q}{\partial\beta}\frac{\partial U}{\partial\beta} + C_{q,U}\left(\frac{\partial U}{\partial\beta}\right)^2. \quad (62)$$

The coefficients $A_{q,U}$, B_U and $C_{q,U}$ are positive at the critical point, since $q > 0$ and $U \to \sqrt{2}$ [39]: $A_c = \frac{8}{q}$, $B_c = 32\sqrt{2}$, and $C_c = 40q$.

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